Analysis of Waveguide Modes by Standing-Wave Pattern Measurements*

H. B. DAVE†, STUDENT MEMBER, IRE

Summary—A method of analyzing a multimode transmission system is described, which is based upon the measurement of mean-square electric field along a line parallel to the waveguide axis. Although the analysis given is for a rectangular waveguide, the method has the advantage that it can readily be adapted to all types of transmission structures.

Introduction

An analysis of the complex modal amplitudes on waveguides, based upon measurement of the transverse electric field, has been made by Forrer and Tomiyasu.[1] Since their method is based upon a numerical Fourier analysis, it is not suitable for transmission structures which involve nonsinusoidal transverse field functions. The method described in the present paper is free from this limitation. It depends on the fact that the various propagating modes, in general, have different phase velocities. Due to this dispersion, the electric field measured along a line parallel to the waveguide axis will show a modulation effect. From the numerical Fourier transform analysis of the resultant field pattern, one can arrive at values of various coefficients for several modal amplitudes.

Theoretical

Consider the expression for $|E_y|^2$ as a function of z in a rectangular waveguide.

$$E_y{}^l = K_y \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\exp j(\omega t - \beta_l z + \delta_l)$$

where

$$K_y = \frac{-j\beta_l C}{h^2}\cdot\frac{n\pi}{b} \text{ for TM modes,}$$

$$= \frac{-j\omega\mu C}{h^2}\cdot\frac{m\pi}{a} \text{ for TE modes;}$$

$E_y{}^l$ = y component of the electric field for a particular mode l;

β_l = phase constant

$$= \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \beta_{mm};$$

z = distance along the waveguide axis;

δ_l = initial phase difference.

The expression for β_{mn} applies both to TM_{mn} and TE_{mn} modes, so the phase velocities of these two modes will be same and there will be degeneracy. For other modes, β_l's are not only different but also are not simply related.

In the analysis given below, degenerate modes are designated by a single number per pair, the electric field being the vector sum of the individual fields.

A. Case of Arbitrary Reflection

At a particular transverse site the y components of incident electric fields are given by

$$\begin{aligned} {}^iE_{y_1} &= A_1 \exp j(\omega t - \beta_1 z + \delta_1) \\ {}^iE_{y_2} &= A_2 \exp j(\omega t - \beta_2 z + \delta_2) \\ &\cdots\cdots\cdots \\ {}^iE_{yl} &= A_l \exp j(\omega t - \beta_l z + \delta_l). \end{aligned} \tag{1}$$

The reflected fields are likewise given by

$$\begin{aligned} {}^rE_{y_1} &= B_1 \exp j(\omega t + \beta_1 z - \delta_1') \\ {}^rE_{y_2} &= B_2 \exp j(\omega t + \beta_2 z - \delta_2') \\ &\cdots\cdots\cdots \\ {}^rE_{yl} &= B_l \exp j(\omega t + \beta_l z - \delta_l'). \end{aligned} \tag{2}$$

The factors $A_1, A_2, \cdots, B_1, B_2$, etc., consist of multiplication of a weighing factor depending upon the probe position and the type of mode and the maximum amplitude of the electric field for the particular mode. It is important that none of these weighing factors be zero, otherwise that particular mode for which the weighing factor is zero will not be detected. If the mode patterns are known, one can always select an axis for probing which will give a reasonable amplitude to each of the modes of interest; *e.g.*, for a rectangular waveguide supporting TE_{10}, TE_{20} and TE_{30} modes, one should avoid locating the probe axis at a distance of $a/2$, $a/3$, $2a/3$ from the narrow face of the waveguide. If the mode patterns on a new waveguide structure are totally unknown, two or three series of probing in the longitudinal direction will help in locating the most suitable axis. Here lies the main advantage of this method, for at least something can be found out about the unknown mode patterns on a new waveguide structure, though actual mode ratios cannot be found.

Equations similar to (1) and (2) can be written for the x component of the electric field and similar analysis can be carried through.

* Received January 25, 1962; revised manuscript received March 8, 1962.

† Department of Electronics and Electrical Communication Engineering, Indian Institute of Technology, Kharagpur, W. Bengal, India.

[1] M. P. Forrer and K. Tomiyasu, "Determination of higher order propagating modes in waveguides," *J. Appl. Phys.*, vol. 29, pp. 1040–1045; July, 1958.

Total electric field in the y direction is given by

$$E_y = \sum {}^iE_{yl} + \sum E_{yl}$$
$$= e^{j\omega t}\begin{bmatrix} A_1 \exp - j(\beta_1 z - \delta_1) + B_1 \exp j(\beta_1 z - \delta_1') \\ + A_2 \exp - j(\beta_2 z - \delta_2) + B_2 \exp j(\beta_2 z - \delta_2') \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ + A_l \exp - j(\beta_l z - \delta_l) + B_l \exp j(\beta_l z - \delta_l') \end{bmatrix}. \quad (3)$$

Since microwave crystal detectors used for such probe measurements are usually very nearly square-law detectors, it is desirable to find the square of $|E_y|$.

$|E_y|^2 = E_y^* \times E_y$, where E_y^* denotes the complex conjugate of E_y.

$$\therefore\ |E_y|^2 = \sum_{r,s=1}^{l} \{2A_rB_s \cos[(\beta_r + \beta_s)z - (\delta_r + \delta_s')]$$
$$+ A_rA_s \cos[(\beta_r - \beta_s)z - (\delta_r - \delta_s)]$$
$$+ B_rB_s \cos[(\beta_r - \beta_s)z - (\delta_r' - \delta_s')]\}. \quad (4)$$

From this equation it will be seen that there is a constant component in the expression for $|E_y|^2$ vs z, moreover there are components varying at a rate of $2\beta_r$, $2\beta_s$, $(\beta_r+\beta_s)$ and $(\beta_r-\beta_s)$ radians/cm. The amplitude coefficient of these components are combinations of $A_1, A_2, \cdots, A_l, B_1, B_2, \cdots, B_l$ taken two at a time.

Of course, (4) reduces to the conventional expression for standing waves in a single-mode waveguide if we make all the coefficients except A_1 and B_1 zero. Thus

$$|E_y|^2 = A_1^2 + B_1^2 + 2A_1B_1 \cos\delta_1' \cdot \cos 2\beta_1 z$$
$$+ 2A_1B_1 \sin\delta_1' \sin 2\beta_1 z$$
$$= K^2(1 + |\rho|^2) + 2K^2|\rho| \cos(\beta_1 z + \phi). \quad (5)$$

B. Case of Perfectly Matched Waveguide

For a perfectly matched waveguide, $B_1, B_2, \cdots, B_l$ are all zero and (4) becomes

$$|E_y|^2 = \sum_{r,s=1}^{l} \{A_rA_s \cos(\delta_r - \delta_s)\cos(\beta_r - \beta_s)z$$
$$+ A_rA_s \sin(\delta_r - \delta_s)\sin(\beta_r - \beta_s)z\}. \quad (6)$$

This equation shows that $|E_y|^2$ consists of a constant component and components varying periodically with respect to z. Thus the periodic variations with respect to z give us the key to determination of the relative amplitudes of the component modes that are present.

C. Method of Calculating the Coefficients

Eq. (4) can be shown to be equivalent to

$$|E_y|^2 = f(z) = P_0 + \sum_j (P_j \cos\bar{\beta}_j z + Q_j \sin\bar{\beta}_j z) \quad (7)$$

where $\bar{\beta}_j$ is of the form $(\beta_r \pm \beta_s)$, and coefficients P_0, P_j and Q_j are functions of A_s, B_s, δ_s and δ_s'.

The $\bar{\beta}_j$ is shown from the properties of the waveguide. Thus if we find P_0, P_j and Q_j from $|E_y|^2 = f(z)$, then we can find A_l, B_l, δ_l and δ_l'. In some cases knowing even P_0, P_j and Q_j may be sufficient.

The situation is analogous to the case of a complex waveform in the time domain which is to be analyzed into various frequency components. If the given function is periodic, then Fourier series expansion may be used. If the given function is aperiodic, then using the Fourier transform we can get the corresponding function in the frequency domain.

In the case under consideration, $\bar{\beta}_j$'s are not harmonically related; the function $f(z)$ is not periodic but rather "almost periodic."[2] This "almost periodicity" means that given an arbitrarily small number ϵ_1, we can always find a period z_1 such that $|f(z+z_1) - f(z)| \le \epsilon_1$ or the function will very approximately be repeated after a certain period z_1. For example, referring to Fig. 3, the pattern is, to the first approximation, repeated after about 7 cm. By using the Fourier transform, one should be able to arrive at various frequency components; the only difficulty is that integration from $-\infty$ to $+\infty$ would have to be obtained. This requires that $f(z)$ be defined between these limits. This, however, is not possible, as $f(z)$ is determined experimentally for a very limited range of z. For practical purposes, a method which gives results with an accuracy greater than or equal to that of experimental readings will be sufficient. One such method is given by Lanczos.[3]

The method is essentially a numerical application of the Fourier transform, using equidistant data. The total number of observational data is assumed to be an odd number $2N+1$. The origin is chosen at $(N+1)$th reading. Observations are made at an interval of ζ cm. A new variable $z' = z/\zeta$ is formed. Thus observations are made at

$z_k' = 0, \pm 1, \pm 2, \cdots, \pm N$. The readings at these points are denoted by $f_k = f(z_k')$.

$$f(z') = \sum_{\alpha=0,1,2,\cdots} (P_\alpha \cos\theta_\alpha z' + Q_\alpha \sin\theta_\alpha z') \quad (8)$$

where $\theta_\alpha = \bar{\beta}_\alpha \cdot \zeta$ and $\theta_0 = 0$. The restriction on ζ is given by

$$\theta_\alpha < \pi,$$
$$i.e.,\ \zeta < \pi/\bar{\beta}_\alpha \quad (9)$$

which is equivalent to applying the "sampling theorem" of Information Theory to this case.

Let $\theta_\alpha = \pi/N \cdot t_\alpha$ where $0 < t_\alpha < N$.

Let $U_k = f_k + f_{-k}$ and $V_k = f_k - f_{-k}$, $k = 0, 1, 2, \cdots, N$.

As the number of observations cannot be expected to be large, the data has to be modified before applying

[2] H. Bohr, "Almost Periodic Functions," Chelsea Publishing Co., New York, N. Y., pp. 32–33; 1951.

[3] C. Lanczos, "Applied Analysis," Prentice-Hall, Inc., New York, N. Y., p. 268; 1956.

Fourier sums. This can be done by multiplying U_k and V_k by a weighting factor

$$\sigma_k = \frac{\sin(k\pi/N)}{k\pi/N} \quad i.e., \quad \overline{U}_k = U_k \cdot \sigma_k, \quad \overline{V}_k = V_k \cdot \sigma_k.$$

$\overline{U}_k$ and $\overline{V}_k$ are transformed into c_k and d_k, where

$$c_k = \sum_{\alpha=0}^{N}{}' \, \overline{U}_k \cos\frac{\pi}{N}\alpha k$$

$$d_k = \sum_{\alpha=1}^{N-1} \overline{V}_k \sin\frac{\pi}{N}\alpha k. \tag{10}$$

Symbol $\sum'$ means that $\overline{U}_0$ and $\overline{U}_N$ enter the process with half weight only.

These c_k's and d_k's can be looked upon as a "line spectrum" corresponding to integer values of $t=k$ of the continuous parameter t_α. The wave number $\bar{\beta}_\alpha$ will generally not correspond to an integer value of t_α, but it will generally lie between $t=i$ and $t=i+1$.

Let t_α, corresponding to $\bar{\beta}_\alpha$, be equal to $i+\psi$ where i is an integer and ψ a pure fraction. Then the peak ordinate c_μ in the spectrum of c_k vs t_α becomes

$$c_\mu = c_i + \psi/4[c_{i+1} - c_{i-1}]. \tag{11}$$

The coefficient P_α in (8) is given by

$$P_\alpha = \frac{c_\mu}{N} \times 1.6963. \tag{12}$$

A similar procedure is used to find Q_α by replacing c_α with d_α. Once P_α and Q_α are known it remains to determine the complex mode coefficients $A_l e^{j\delta_l}$ and $B_l e^{-j\delta_l'}$. These coefficients can be found by writing the explicit forms of P_α and Q_α in terms of A_l, B_l, δ_l and δ_l' and then solving the resulting simultaneous equations for A_l, B_l, δ_l and δ_l'.

As an example consider a two-mode system

$$A_1^2 + B_1^2 + A_2^2 + B_2^2 = P_0 \tag{13a}$$

$$2A_1B_2\cos\delta_2' + 2A_2B_1\cos(\delta_1' + \delta_2) = P_1 \tag{13b}$$

$$2A_1A_2\cos\delta_2 + 2B_1B_2\cos(\delta_1' - \delta_2') = P_2 \tag{13c}$$

$$2A_1B_1\cos\delta_1' = P_3 \tag{13d}$$

$$2A_2B_2\cos(\delta_2 + \delta_2') = P_4 \tag{13e}$$

$$2A_1B_2\sin\delta_2' + 2A_2B_1\sin(\delta_1' + \delta_2) = Q_1 \tag{13f}$$

$$2A_1A_2\sin(-\delta_2) + 2B_1B_2\sin(\delta_1' - \delta_2') = Q_2 \tag{13g}$$

$$2A_1B_1\sin\delta_1' = Q_3 \tag{13h}$$

$$2A_2B_2\sin(\delta_2 + \delta_2') = Q_4. \tag{13i}$$

If the load is almost or perfectly matched, then all coefficients except P_0, P_2, Q_2 approach zero and we have

$$A_1^2 + A_2^2 = P_0 \tag{14a}$$

$$2A_1A_2\cos\delta_2 = P_2 \tag{14b}$$

$$2A_1A_2\sin(-\delta_2) = Q_2 \tag{14c}$$

$$\therefore\ 2A_1A_2 = \sqrt{P_2^2 + Q_2^2} \tag{15a}$$

and

$$\tan\delta_2 = -\frac{Q_2}{P_2}. \tag{15b}$$

Eqs. (15a) and (15b) will give two values of A_1/A_2, which are reciprocals of each other. Let $A_1/A_2 = s$ or $1/s$ where $s>1$. This ambiguity is removed by noting that due to attenuation in the guide, A_2 reduces more quickly with distance than A_1 does. Thus by noting the value of s at sections M and N and s' at sections M' and N' in Fig. 2, we can find value of A_1/A_2. Thus

$$\text{if } s' > s, \qquad A_1 > A_2 \quad \text{or} \quad \frac{A_1}{A_2} > 1$$

$$\text{if } s > s', \qquad A_2 > A_1 \quad \text{or} \quad \frac{A_1}{A_2} < 1.$$

For degenerate modes, the procedure suggested by Forrer and Tomiyasu[1] will be useful.

Experimental Work

The experimental work was carried out at about 9300 Mc. Rectangular waveguide was used, the dimensions being chosen so that only TE_{10} and TE_{20} modes propagated. The TE_{20} mode was excited by putting a wire probe oriented along y axis in the center of the waveguide and adjusting its length to obtain an arbitrary division of energy between the modes. For the probe depth used in the experiment described below, it turned out that approximately equal amplitudes of the two modes were launched in the waveguide.

The basic measurements were made with an HP-444A untuned probe which was inserted successively in a series of closely spaced holes in the broad face of the measuring guide (see Fig. 1). A matched load was prepared by inserting a wooden pyramid painted with carbon resistant paint in a waveguide. Insertion of a dielectric slab in front of the matched load produced an arbitrary reflecting load. Typical patterns of $|E_y|^2$ vs z which were obtained with these loads, together with that obtained with a short circuit, are shown in Figs. 2–4. From the set of readings plotted in Fig. 3, the following values of P_1, etc., were calculated.

$$P_1 = 2.425 \qquad Q_1 = 1.154$$

$$P_2 = 1.274 \qquad Q_2 = -2.48$$

$$P_3 = 0.969 \qquad Q_3 = -0.747$$

$$P_4 = 0.321 \qquad Q_4 = -1.188$$

$$A_1 = 0.783 \qquad A_2 = 0.785$$

$$B_1 = 0.781 \qquad B_2 = 0.783.$$

The analysis of the curve shown in Fig. 2 is quite simple and the results obtained are

$$A_1 = 1.32 \qquad A_2 = 1.00.$$

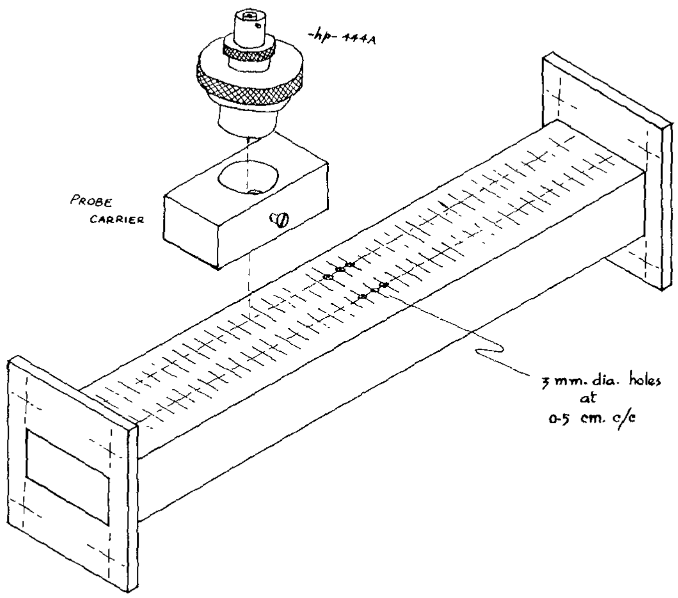


Fig. 1—Isometric view of the measurement waveguide.

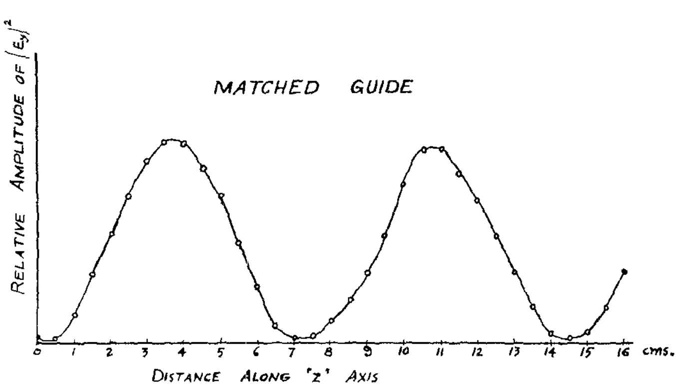


Fig. 2—$|E_y|^2$ vs z pattern for matched load. $\beta_1-\beta_2=0.908$ radian/cm.

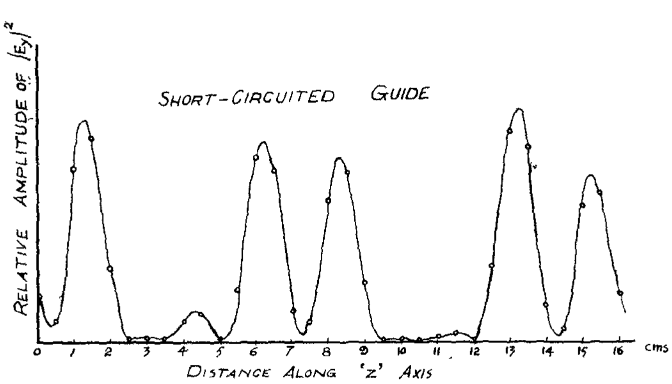


Fig. 3—$|E_y|^2$ vs z pattern for short-circuited waveguide. $\beta_1=1.783$, $\beta_2=0.874$.

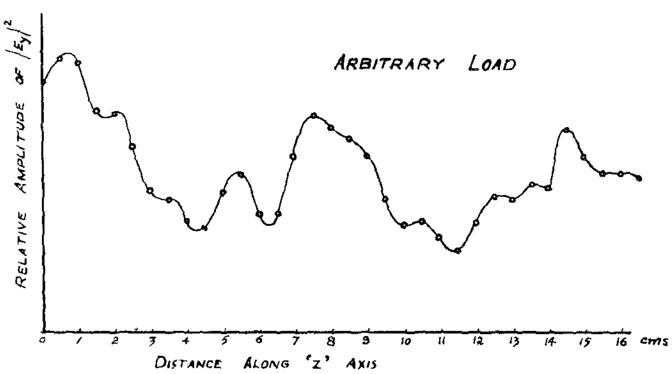


Fig. 4—$|E_y|^2$ vs z pattern for an arbitrary load.

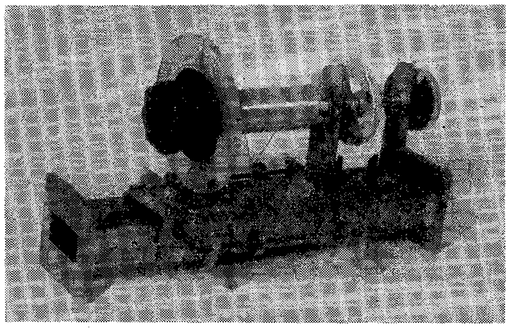


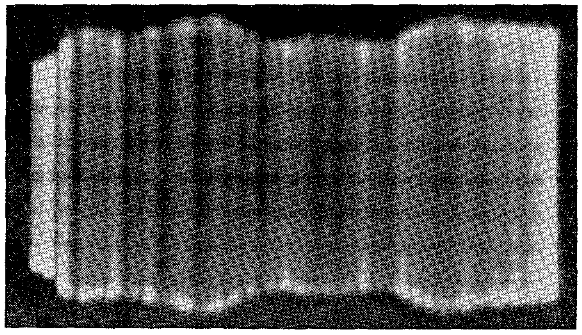
Fig. 5—Movable dielectric slab phase-shifter.

The analysis of the $|E_y|^2$ vs z pattern for the arbitrary load, Fig. 4, gave the following results

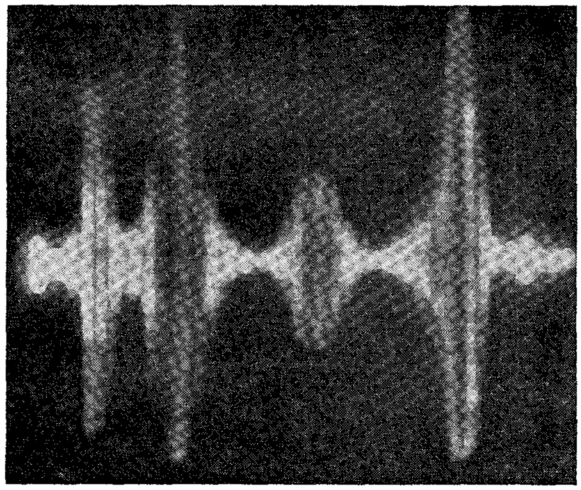
$$A_1 = 2.31 \qquad A_2 = 0.71$$
$$B_1 = 0.168 \qquad B_2 = 0.42.$$

It may be seen from the above results that the condition of match can be identified by the fact that a simple sinusoidal pattern exists in the longitudinal direction. This suggests that for some purposes, where a computation of amplitudes is not necessary but a condition of match is to be observed, an oscilloscopic representation of the data might be used. A convenient way of obtaining such data is to use a fixed probe position and shift the phase between various modes.

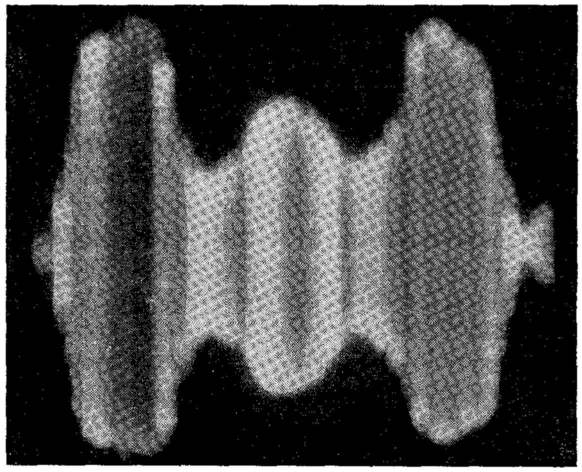
A movable dielectric slab phase-shifter, Fig. 5, was used for changing the phase between the two modes. A potentiometer was attached to the drum which controlled the position of the slab in the waveguide. The voltage, determined by the setting of the potentiometer, gave the horizontal deflection on a slow-speed oscilloscope. The output from the probe, after amplification, was applied to the vertical deflection plates. Thus the pattern of $|E_y|^2$ vs slab position was obtained at a fixed probe position. Such a pattern was recorded for a matched load condition [see Fig. 6(a)]. This pattern could subsequently be used as a comparison standard to match unknown loads. Thus matching for two modes in a waveguide is made easier.



(a)



(b)



(c)

Fig. 6—(a) $|E_y|^2$ vs slab position for matched load. (b) $|E_y|^2$ vs slab position for short-circuited waveguide. (c) $|E_y|^2$ vs slab position for an arbitrary load.

Conclusion

A method of determining the mode content in a general waveguide has been described. The usefulness of the procedure lies in the fact that it can be used with any kind of dispersive transmission system. The method does not finally solve the problem of multimode analysis, but it is felt that it gives a better understanding about the operation of a multimode waveguide. The limitation of the procedure lies in the fact that an enormous amount of numerical calculation is involved even with a small number of modes. By changing the phase between the various propagating modes, *e.g.*, by a movable dielectric phase-shifter, the pattern of $|E_y|^2$ vs $\Delta\delta$, at a fixed probe position, can be displayed on an oscilloscope which can at least tell us whether the waveguide is matched or not.

Acknowledgment

The author would like to thank Prof. W. L. Emery, Technical Cooperation Mission Visiting Professor, and Prof. G. S. Sanyal for their able guidance and supervision of the work. Dr. H. Rakshit, Head of the Department of Electronics and Electrical Communication Engineering, took keen interest in the work and arranged for the fabrication of the waveguide components. The author sincerely thanks him.